A Simple Equation to Represent Dimensionless Unit Hydrographs

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by

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Introduction

The purpose of this paper is to present a general equation for curvilinear dimensionless unit hydrographs. This equation is of a simple form, is a function of one parameter, and produces a broad range of dimensionless unit hydrograph shapes.

SCS Dimensionless Unit Hydrograph and Peak Rate Equation

The Soil Conservation Service (SCS) recommends the use of their curvilinear or triangular dimensionless unit hydrographs for estimating the peak rate and time distribution of runoff from small watersheds. Figure 1 illustrates the SCS dimensionless unit hydrographs. The SCS determined the shape of the curvilinear dimensionless unit hydrograph by analyzing a large number of watersheds from a variety of sites throughout the U.S. The base time, T_b, of the equivalent triangular dimensionless unit hydrograph was determined by setting the area under the curvilinear hydrograph equal to the area under the equivalent triangular hydrograph.

The SCS also recommends the use of the following equation for estimating the peak rate of runoff:

$$q_p = 484*A*Q/t_p \qquad (1)$$

where
q_p = peak runoff rate [cfs]
A = watershed area [mi^2]
Q = excess precipitation [in]
t_p = time to peak runoff [hrs]
484 = SCS peak rate factor [cfs/(mi^2*in/hr)]

Equation 1 applies to the standard SCS curvilinear and triangular dimensionless unit hydrographs shown in Figure 1. Several researchers have indicated that the SCS peak rate factor of 484 is too large for flatwoods watersheds. Among these are the SCS, USDA-ARS, and the University of Florida. Flatwoods watersheds are common in central and southern Florida, and are characterized by very flat slopes, extremely permeable sandy soils, high water tables, and scattered wetlands. The SCS (1957) recommends a peak rate factor of 300 be used for flat-swampy areas. The University of Florida (1986) has found that a peak rate factor of 75-100 is appropriate for flatwoods watersheds. However, the University of Florida's results were based on data from 5 watersheds with storm events having less than a 5-yr return period. More extreme rainfall events may have produced higher peak rate factors.

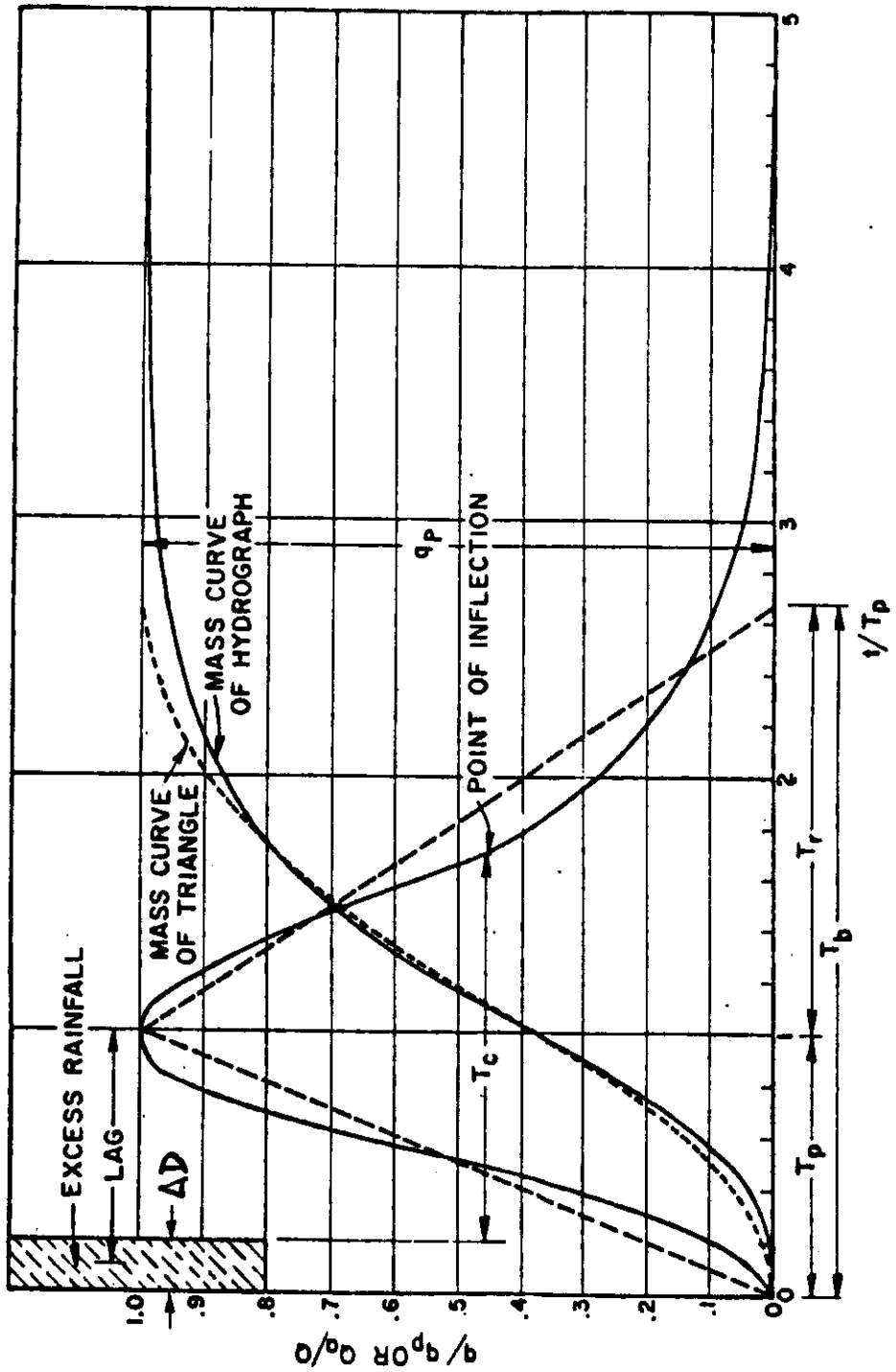

Figure 1 SCS Curvilinear and Equivalent Triangular Dimensionless Unit Hydrographs

If a peak rate factor other than 484 is to be used for a runoff analysis, then a dimensionless unit hydrograph other than that of Figure 1 must be used. Failure to use the correct dimensionless unit hydrograph will result in an incorrect computation of the runoff volume.

The SCS derived the peak rate factor of 484 by using the geometric relationships of the equivalent triangular dimensionless unit hydrograph. Although this derivation is straightforward, it does not illustrate the important relationship between the area under the dimensionless unit hydrograph and the peak rate factor. The following discussion illustrates this relationship.

The volume, V, of runoff from a watershed with area, A, for an excess precipitation amount, Q, is:

$$V = A*Q \tag{2}$$

This runoff volume is distributed over time via a runoff hydrograph. The area under the runoff hydrograph must equal the runoff volume.

$$\int_0^\infty q_t*dt = V = A*Q \tag{3}$$

where q_t = runoff rate at time t

If the runoff hydrograph is made dimensionless by dividing q_t by q_p (the peak runoff rate), and t by t_p (the time to peak), then the following relationship results:

$$\int_0^\infty q_t*dt = q_p*t_p \int_0^\infty (q_t/q_p)*d(t/t_p) \tag{4}$$

Let

$$x = \int_0^\infty (q_t/q_p)*d(t/t_p) \tag{5}$$

Note that x is the dimensionless area under the dimensionless unit hydrograph. Combining Equations 3,4 and 5 yields:

$$q_p*t_p*x = A*Q \tag{6}$$

To compute the peak discharge in cfs with $[A]=mi^2$, [Q]=inches, and $[t_p]$=hours, the conversion factor, $645.33[cfs/(mi^2*in/hr)]$ must be applied. Rearranging Equation 6 and incorporating this conversion factor yields:

$$q_p = [645.33/x]*A*Q/t_p \tag{7}$$

where $B = [645.33/x]$ = the peak rate factor (8)

For the SCS dimensionless unit hydrograph shown in Figure 1, the dimensionless area, x=1.33, and the corresponding peak rate factor, B = 645.33/1.33 = 484. The peak rate factor depends only on the area under the dimensionless unit hydrograph (DUH), and vice-versa. Therefore, changing the peak rate factor must result in a new area and shape of the DUH.

General Curvilinear Dimensionless Unit Hydrograph (GCDUH)

The Nash Model (1957) is a conceptual model of a catchment having the same hydrologic response as a series of n cascaded linear reservoirs, each having the same storage coefficient, K. The equation of continuity for a linear reservoir is given by:

$$I - Q = K*dQ/dt \tag{9}$$

For an instantaneous inflow, I, input into the first linear reservoir, the unit outflow can be represented by:

$$Q_1(t) = (1/K)*EXP(-t/K) \tag{10}$$

This outflow is used as the inflow for the second linear reservoir. The outflow from the second linear reservoir is represented by:

$$Q_2(t) = (t/K^2)*EXP(-t/K) \tag{11}$$

This outflow is used as the inflow for the third linear reservoir. Repeating the process for n linear reservoirs yields the Nash Model:

$$Q_n(t) = (t/K)^{n-1}*EXP(-t/K)/\{K*(n-1)!\} \tag{12}$$

The parameters n and K can be varied to produce a variety of hydrograph shapes. A closer look at Equation 11 indicates that the rather ominous looking equation can be simplified by noting the constant terms.

Let

$$a = (1/K)^n/(n-1)! = \text{constant} \tag{13}$$
$$b = n-1 = \text{constant} \tag{14}$$
$$c = 1/K = \text{constant} \tag{15}$$
$$t/t_p = t \tag{16}$$
$$q/q_p = Q_n \tag{17}$$

So,

$$[q/q_p] = a*[t/t_p]^b*EXP(-c*[t/t_p]) \tag{18}$$

But, for the dimensionless unit hydrograph,

$$q/q_p = 1 \text{ @ } t/t_p = 1; \text{ which yields: } a = EXP(c) \tag{19}$$

and,

$$[q/q_p]' = 0 \text{ @ } t/t_p = 1; \text{ which yields: } b = c \tag{20}$$

Substituting Equations 19 and 20 into Equation 18 yields:

$$[q/q_p] = \{[t/t_p]*EXP[1-t/t_p]\}^c \tag{21}$$

Equation 21 is a general equation for the curvilinear dimensionless unit hydrograph (GCDUH), and is a function of only one parameter, c. This parameter, or GCDUH exponent, can be related to the area under the GCDUH.

The area under the GCDUH is:

$$x = \int_0^{\infty} \{[t/t_p]*EXP(1-[t/t_p])\}^c * d[t/t_p] \qquad (22)$$

Integrating yields: $$x = EXP(c)*\Gamma(c+1)/c^{c+1} \qquad (23)$$

An explicit equation for c as a function of x is not possible to obtain. A 4-th order polynomial was fit to Equation 23 to provide a working relationship for the GCDUH exponent as a function of the peak rate factor. This relationship is valid for peak rate factors up to 600 and is shown in Figure 2.

The GCDUH (Equation 21) and the equation from Figure 2 can be used to determine the shape of the dimensionless unit hydrograph for a variety of peak rate factors. Figure 3 illustrates the variety of DUH shapes that are generated for 5 peak rate factors. Figure 4 illustrates the same 5 GCDUH's transformed into unit hydrographs for a one square mile drainage area having a one hour time to peak. These figures illustrate only a few of the possible unit hydrograph shapes that can be defined by the GCDUH.

The use of the GCDUH and Figure 2 to determine the DUH is explained via the following example. Consider the peak rate factor of 484. Equation 8 can be solved for the dimensionless area, x; (x = 645.33/484 = 1.33). The polynomial from Figure 2 converts this area to a GCDUH exponent of 3.70. Thus, the entire dimensionless hydrograph is defined by Equation 21 with c=3.70. Figure 5 illustrates a comparison of the GCDUH with c=3.70 (B=484) to the breakpoint data given by the SCS for their curvilinear DUH. The GCDUH fits the SCS data relatively well. The coefficient of determination (r^2) of 0.9941 indicates that only .59% of the DUH variation is unexplained by the GCDUH.

Summary

The General Curvilinear Dimensionless Unit Hydrograph (GCDUH) given by Equation 21 was derived by: (1) simplifying the terms of the Nash Model, (2) normalizing the time and discharge variables, and (3) noting the properties of the curvilinear dimensionless unit hydrograph. The GCDUH equation can be used to easily compute the ordinates of the DUH for peak rate factors up to 600. The simple equation describes the entire DUH and eliminates the need to interpolate between breakpoint data. The equation closely approximates the standard SCS curvilinear DUH and also fits a broad range of unit hydrograph shapes.

References Cited

1. Soil Conservation Service, 1957. National Engineering Handbook, Section 4, Hydrology. U.S. Department of Agriculture, Washington D.C.

2. Capece, J.C., K.L. Campbell, and L.B. Baldwin, 1986. Estimation of Runoff Peak Rates and Volumes from Flatwoods Watersheds. University of Florida, Gainesville. Report submitted to the South Florida Water Management District.

GCDUH EXPONENT VS. PEAK RATE FACTOR

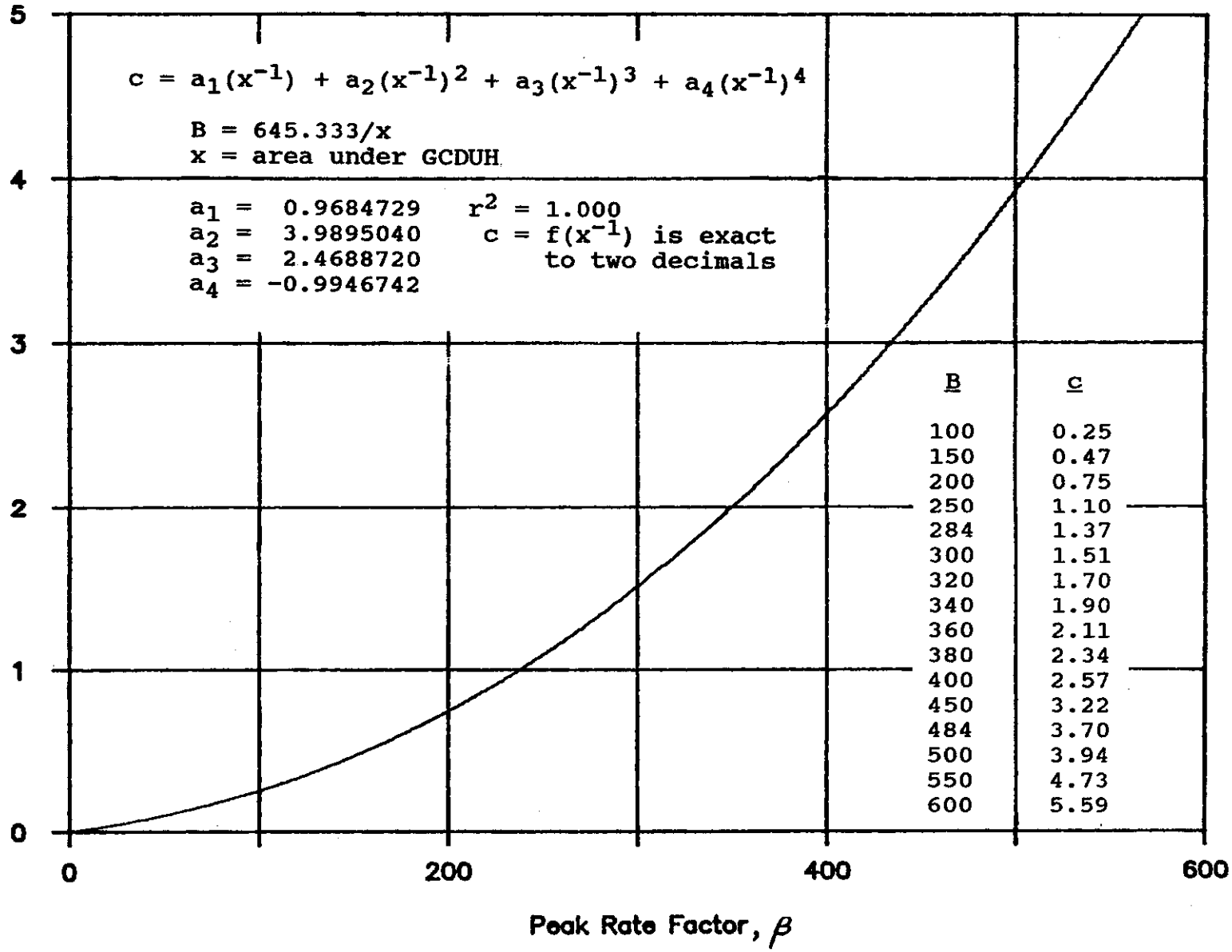

Figure 2

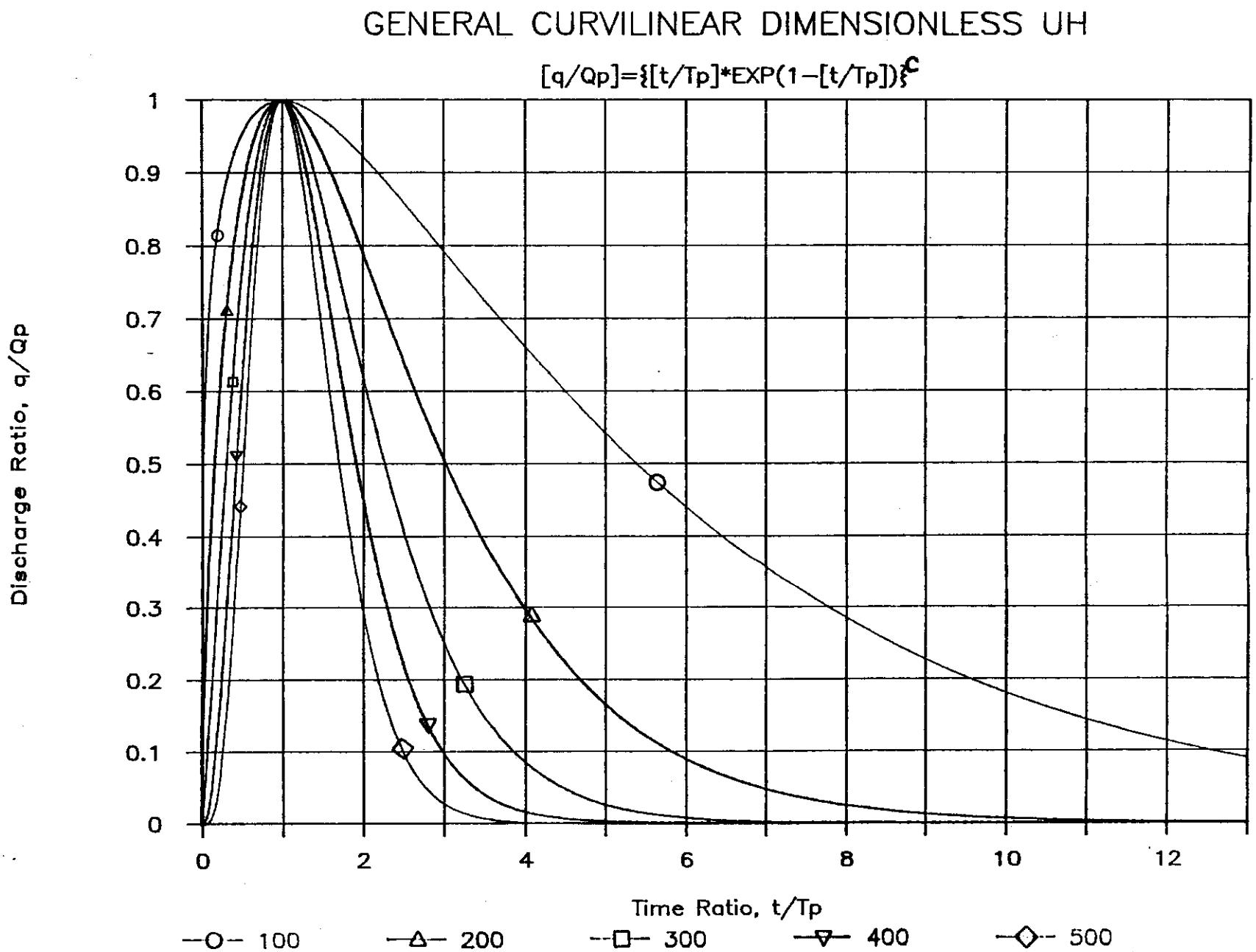
GENERAL CURVILINEAR DIMENSIONLESS UH
[q/Qp]={[t/Tp]*EXP(1-[t/Tp])}^C
Discharge Ratio, q/Qp
Time Ratio, t/Tp
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
2
4
6
8
10
12
100
200
300
400
500

Figure 3

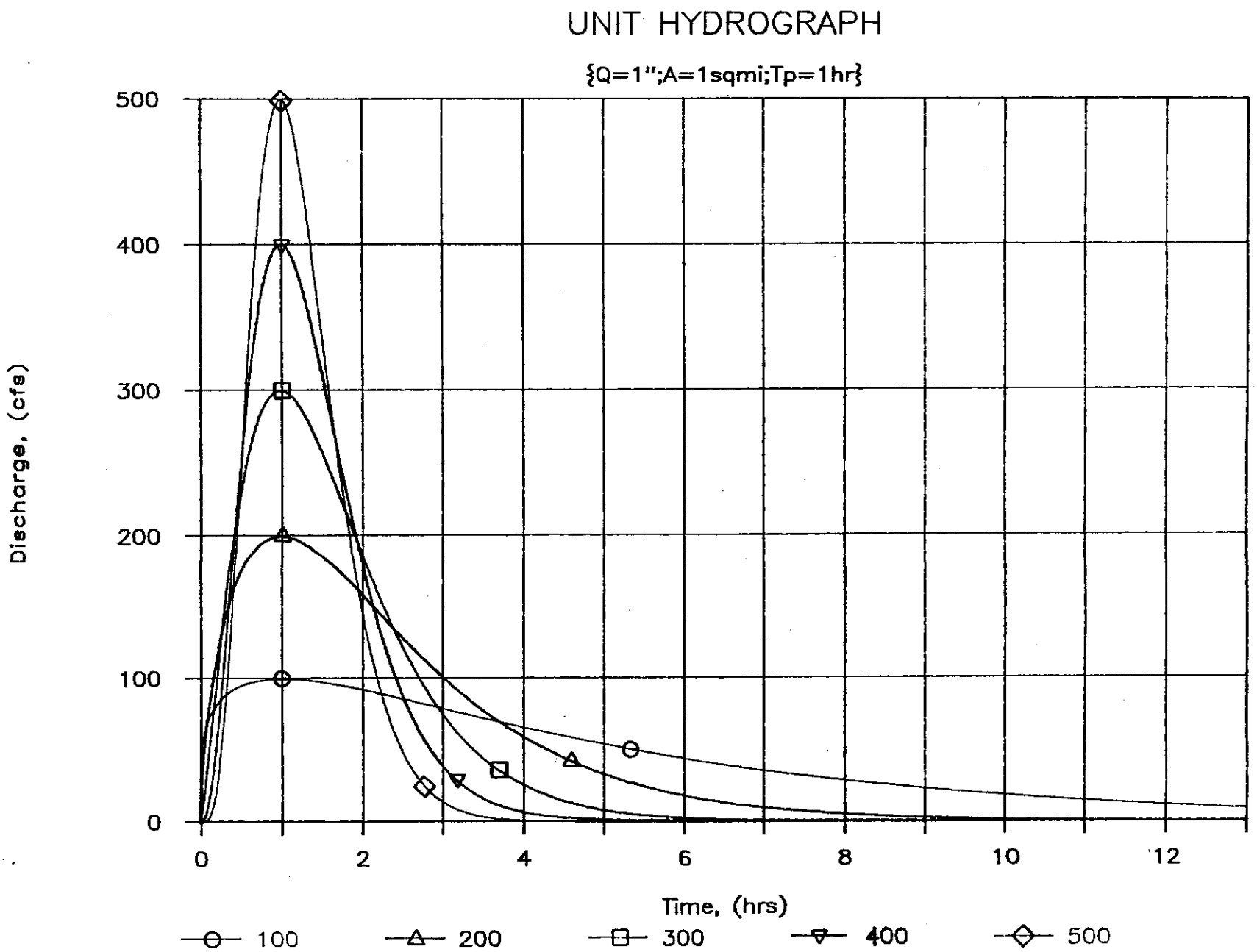
UNIT HYDROGRAPH
{Q=1";A=1sqmi;Tp=1hr}
Discharge, (cfs)
500
400
300
200
100
0
0
2
4
6
8
10
12
Time, (hrs)
100
200
300
400
500

Figure 4

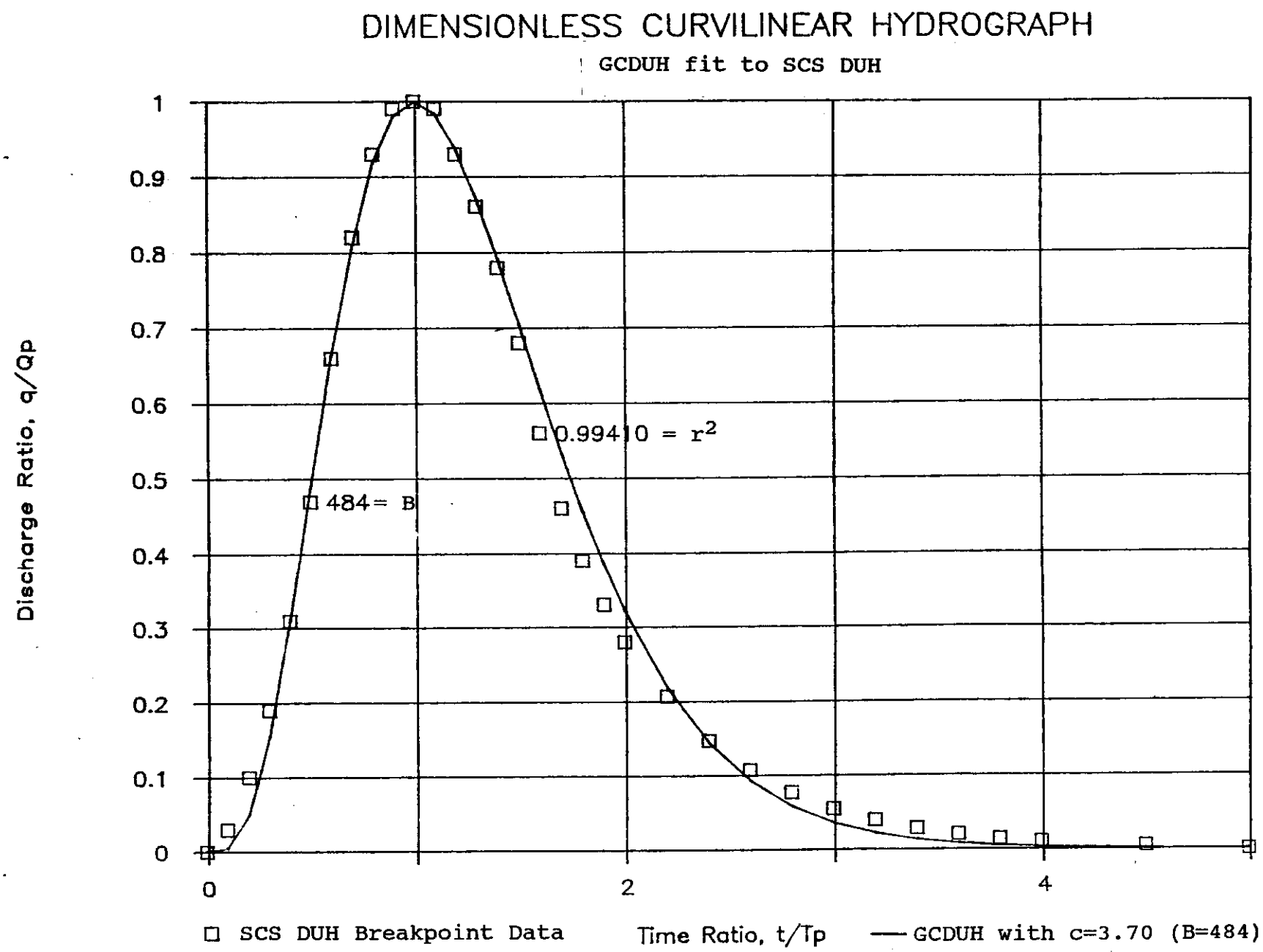
DIMENSIONLESS CURVILINEAR HYDROGRAPH
GCDUH fit to SCS DUH
Discharge Ratio, q/Qp
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.99410 = r2
484= B
0
2
4
SCS DUH Breakpoint Data
Time Ratio, t/Tp
GCDUH with c=3.70 (B=484)

Figure 5